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Symmetries and motions in manifolds

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In these lectures the relations between symmetries, Lie algebras, Killing vectors and Noether's theorem are reviewed. A generalisation of the basic ideas to include velocity-dependent co-ordinate transformations naturally leads to the concept of Killing tensors. Via their Poisson brackets these tensors generate an a priori infinite-dimensional Lie algebra. The nature of such infinite algebras is clarified using the example of flat space–time. Next the formalism is extended to spinning space, which in addition to the standard real co-ordinates is parametrised also by Grassmann-valued vector variables. The equations for extremal trajectories ("geodesics") of these spaces describe the pseudo-classical mechanics of a Dirac fermion. We apply the formalism to solve for the motion of a pseudo-classical electron in Schwarzschild space–time.

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1. Motions of scalar points in curved space–time

1.1. INTRODUCTION

In the following we connect a number of old and venerable topics related to symmetries and conservation laws, such as Lie algebras and Noether's theorem, with differential geometric structures like Lie derivatives and Killing vectors. Although most of the basic ideas are well known [#1], we present extensions and generalisations of interest in the description of certain physical systems; in particular we apply our methods to study the motion of spinning particles in a curved space–time.

According to Einstein's equivalence principle the world line of a massive scalar point particle in curved space–time is a time-like geodesic, described by the

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[#1] See, for example, refs. [1–3].

equation

$$D^2x^\mu/D\tau^2 = \ddot{x}^\mu + \Gamma_{\lambda\nu}{}^\mu \dot{x}^\lambda \dot{x}^\nu = 0, \tag{1}$$

with overdots denoting proper time derivatives. Its time-like nature is expressed by the condition

$$\left(\frac{\mathrm{d}s}{\mathrm{d}\tau}\right)^2 = g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = -c^2 < 0, \tag{2}$$

where the universal constant c (the light velocity) is a real number. In the following we usually take $c = 1$ and consider particles of unit mass, but occasionally we re-instate the explicit mass dependence when this is physically relevant.

Note that eq. (1) implies that the acceleration due to gravity is quadratic in the four-velocity, whereas according to the Lorentz force law the acceleration of a particle subject to electro-magnetic forces is linear in the four-velocity. For particles coupled to force fields of higher spin one expects the acceleration to depend on higher powers of the four-velocity [4].

The geodesic law of motion (1) can be derived from an action principle. The simplest form of the action is

$$S = \int_1^2 \mathrm{d}\tau \, \tfrac{1}{2} g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu, \tag{3}$$

the stationary points of which are precisely given by eq. (1). Indeed, the general variation of S reads

$$\delta S = \int_1^2 \mathrm{d}\tau \left\{ -\delta x^\mu \, g_{\mu\nu} \frac{D^2 x^\nu}{D\tau^2} + \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\delta x^\mu p_\mu\right) \right\}, \tag{4}$$

where $p_\mu = g_{\mu\nu}\dot{x}^\nu$ is the canonical momentum. Thus δS vanishes for any arbitrary variation of x^μ with fixed end points if and only if the equations of motion (1) are satisfied.

1.2. SYMMETRIES AND NOETHER'S THEOREM

In regard to eq. (4) we can now ask whether there exist variations δx^μ for which $\delta S = 0$ modulo boundary terms even when the equations of motion are *not* satisfied:

$$\delta S = \int_1^2 \mathrm{d}\tau \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\delta x^\mu p_\mu - \mathcal{J}(x, \dot{x})\right). \tag{5}$$

Here we have already split off the total derivative coming from partial integrations in the derivation of expression (4). The quantity $\mathcal{J}(x,\dot{x})$ is obtained from variations δx^μ of the type

$$\delta x^\mu = \mathcal{R}^\mu(x,\dot{x}) = R^\mu(x) + \dot{x}^\nu K_\nu{}^\mu(x) + \tfrac{1}{2}\dot{x}^\nu \dot{x}^\lambda L_{\nu\lambda}{}^\mu(x) + \cdots . \tag{6}$$

We restrict ourselves to variations which depend on the first derivative $\dot{x}^\mu$ only, because the second and higher derivatives can always be rewritten in terms of these modulo the equations of motion (1). Comparing eqs. (4) and (5) one immediately finds

$$\frac{\mathrm{d}\mathcal{J}}{\mathrm{d}\tau} = \mathcal{R}^\mu g_{\mu\nu} \frac{D^2 x^\nu}{D\tau^2} \approx 0, \tag{7}$$

where the last equality holds only upon using the equations of motion. Hence for physical motions the quantities $\mathcal{J}(x, \dot{x})$ are conserved. This is Noether's theorem.

1.3. GENERALISED KILLING EQUATIONS

Assuming that $\mathcal{J}(x, \dot{x})$ can be expanded in the four-velocity as

$$\mathcal{J}(x, \dot{x}) = J^{(0)}(x) + \dot{x}^\mu J^{(1)}_\mu(x) + \tfrac{1}{2}\dot{x}^\mu \dot{x}^\nu J^{(2)}_{\mu\nu}(x) + \frac{1}{3!}\dot{x}^\mu \dot{x}^\nu \dot{x}^\lambda J^{(3)}_{\mu\nu\lambda}(x) + \cdots, \tag{8}$$

we can compare terms with equal powers of the acceleration and velocity on the left- and right-hand side of eq. (7), using the ansatz (6) for δx^μ. This leads first of all to an identification of the co-efficients in the expansions (6) and (8):

$$\begin{aligned} J^{(1)}_\mu(x) &= R_\mu(x), \\ J^{(2)}_{\mu\nu}(x) &= K_{\mu\nu}(x), \\ J^{(3)}_{\mu\nu\lambda}(x) &= L_{\mu\nu\lambda}(x), \quad \text{etc.}, \end{aligned} \tag{9}$$

indicating that all covariant tensors on the right-hand side of eq. (6) should be taken to be completely symmetric. Secondly, the comparison shows that the following differential equations have to be satisfied:

$$J^{(n)}_{(\mu_1 \cdots \mu_n ; \mu_{n+1})} = 0, \tag{10}$$

in which the parentheses denote full symmetrisation over all indices enclosed, with total weight one. Equations (10) constitute a straightforward generalisation of the Killing equation for the isometries of differentiable manifolds. Explicitly

$$J^{(0)}_{,\mu} = 0, \tag{11}$$

$$R_{(\mu;\nu)} = 0, \tag{12}$$

$$K_{(\mu\nu;\lambda)} = 0, \quad \text{etc.} \tag{13}$$

The first equation, eq. (11), implies that $J^{(0)}$ is an irrelevant constant which we ignore from now on. The second equation, eq. (12), is the standard equation for Killing vectors, whilst eq. (13) and its higher-rank counterparts constitute tensorial generalisations of this equation. Therefore one refers to $K_{\mu\nu}$ and higher-rank tensors satisfying eq. (10) as Killing tensors.

1.4. CANONICAL ANALYSIS

In terms of phase-space variables (x^μ, p_μ) the conserved quantities of motion read

$$\mathcal{J}(x,p) = p_\mu R^\mu(x) + \tfrac{1}{2} p_\mu p_\nu K^{\mu\nu}(x) + \frac{1}{3!} p_\mu p_\nu p_\lambda L^{\mu\nu\lambda}(x) + \cdots . \qquad (14)$$

Suppose that there exist n independent Killing vectors $R_a^\mu(x)$, $a = 1, ..., n$. Then a general Killing vector is a linear combination

$$R^\mu[\xi] = \xi^a R_a^\mu, \qquad (15)$$

where the ξ^a constitute a set of n linearly independent parameters. A similar remark holds for Killing tensors. Hence a particular $\mathcal{J}$, eq. (14), is in general specified by the values of these parameters $\xi^A = (\xi^a, ...)$.

Introducing the fundamental Poisson brackets

$$\{x^\mu, p_\nu\} = \delta_\nu^\mu, \qquad (16)$$

we can compute the Poisson bracket of two conserved Noether charges $\mathcal{J}(1) \equiv \mathcal{J}[\xi_1^A]$ and $\mathcal{J}(2) \equiv \mathcal{J}[\xi_2^A]$. The result is

$$\begin{aligned}\{\mathcal{J}(1), \mathcal{J}(2)\} = \; & p_\mu \Big(R^\mu(1)_{;\lambda} R^\lambda(2) - R^\mu(2)_{;\lambda} R^\lambda(1) \Big) \\ & + \tfrac{1}{2} p_\mu p_\nu \Big(K^{\mu\nu}(1)_{;\lambda} R^\lambda(2) - K^{\mu\nu}(2)_{;\lambda} R^\lambda(1) \\ & \quad + 2R^\mu(1)_{;\lambda} K^{\nu\lambda}(2) - 2R^\mu(2)_{;\lambda} K^{\nu\lambda}(1) \Big) \\ & + \frac{1}{3!} p_\mu p_\nu p_\lambda \Big(L^{\mu\nu\lambda}(1)_{;\kappa} R^\kappa(2) - L^{\mu\nu\lambda}(2)_{;\kappa} R^\kappa(1) \\ & \quad - 3K^{\kappa(\mu\nu}(1) R^{\lambda)}(2)_{;\kappa} + 3K^{\kappa(\mu\nu}(2) R^{\lambda)}(1)_{;\kappa} \\ & \quad + 3K_{;\kappa}^{(\mu\nu}(1) K^{\lambda)\kappa}(2) - 3K_{;\kappa}^{(\mu\nu}(2) K^{\lambda)\kappa}(1) \Big) \\ & + \cdots . \end{aligned} \qquad (17)$$

The left-hand side being conserved for physical motions, the right-hand side must again be of the form (14), modulo equations of motion. We now restrict ourselves to off-shell closed algebras, i.e. the case in which the Poisson bracket (17) itself takes the form of a Noether charge without explicit use of the equations of motion. Then

$$\{\mathcal{J}(1), \mathcal{J}(2)\} = \mathcal{J}(3), \qquad (18)$$

where the parameters ξ_3 are bilinear combinations of ξ_1 and ξ_2:

$$\xi_3^A = f^{BCA} \xi_2^B \xi_1^C . \qquad (19)$$

From eqs. (17)–(19) it now follows that

$$[\mathcal{L}_{R^a}(R)]^{b\mu} \equiv R^{b\mu}{}_{;\nu}R^{a\nu} - R^{a\mu}{}_{;\nu}R^{b\nu} = f^{abc}R^{c\mu}, \tag{20}$$

$$[\mathcal{L}_{R^a}(K)]^{i\,\mu\nu} \equiv K^{i\,\mu\nu}{}_{;\lambda}R^{a\lambda} - 2R^{a(\mu}{}_{;\lambda}K^{i\,\nu)\lambda} = t^{aij}K^{j\,\mu\nu}, \tag{21}$$

$$[\mathcal{L}_{R^a}(L)]^{r\,\mu\nu\lambda} \equiv L^{r\mu\nu\lambda}{}_{;\kappa}R^{a\kappa} - 3R^{a(\mu}{}_{;\kappa}L^{r\,\nu\lambda)\kappa} = g^{ars}L^{s\,\mu\nu\lambda}, \quad \text{etc.} \tag{22}$$

Here the symbol $\mathcal{L}$ has been introduced to denote Lie derivatives. In addition, we find relations involving higher-rank Killing tensors of the type

$$[\mathcal{L}_{K^j}(K)]^{i\,\mu\nu\lambda} \equiv K^{i(\mu\nu}{}_{;\kappa}K^{j\,\lambda)\kappa} - K^{j(\mu\nu}{}_{;\kappa}K^{i\,\lambda)\kappa} = c^{ijr}L^{r\,\mu\nu\lambda}, \tag{23}$$

and its further generalisations. Thus the notion of Lie derivative is extended to higher-rank tensors.

Observe that eq. (20) implies that the Killing vectors define an n-dimensional Lie algebra. Equations like (21), (22) then assert that the Killing tensors transform in linear representations of this Lie algebra defined by the structure constants $(t^{aij}, g^{ars}, ...)$. Indeed, the Jacobi identities guarantee that these structure constants realise the Lie algebra via their matrix commutators:

$$\begin{aligned} \left[t^a, t^b\right]^{ij} &= -f^{abc}\, t^{cij}, \\ \left[g^a, g^b\right]^{rs} &= -f^{abc}\, g^{crs}, \quad \text{etc.} \end{aligned} \tag{24}$$

From eq. (23) it follows, that Killing tensors of rank n and m generate Killing tensors of rank $(n + m - 1)$ via their tensorial Lie derivatives. In this way one obtains in principle an infinite-dimensional algebra of conserved quantities, unless the left-hand side of eq. (23) vanishes identically, as might happen in special cases.

1.5. A UNIVERSAL SOLUTION

The generalised Killing equations (10) admit one solution which exists for any arbitrary metric $g_{\mu\nu}$. This solution is generated by the metric itself:

$$K_{\mu\nu}(x) = g_{\mu\nu}(x). \tag{25}$$

It satisfies eq. (10) identically by virtue of the metric postulate $g_{\mu\nu;\lambda} = 0$. The constant of motion constructed from this Killing tensor is the world-line Hamiltonian

$$H(x,p) = \tfrac{1}{2}\, g^{\mu\nu}(x)\, p_\mu p_\nu, \tag{26}$$

which is the generator of proper-time translations. From this solution we can construct a whole tower of higher-order Killing tensors by taking completely symmetrised products of n metric tensors. The corresponding conserved quantities are given by the nth power of the Hamiltonian (26). Clearly, all these Noether charges commute among themselves, and no other solutions are generated by their Poisson brackets. The existence of any further solutions to the generalised Killing equations depends on the specific choice of $g_{\mu\nu}(x)$.

Note, that the method of constructing higher-rank Killing tensors out of products of lower ones works quite generally. Indeed, from any two Killing tensors $J^{(n)}_{\mu_1\cdots\mu_n}$ and $J^{(m)}_{\mu_1\cdots\mu_m}$ one can construct a new Killing tensor

$$J^{(n+m)}_{\mu_1\cdots\mu_{n+m}} = J^{(n)}_{(\mu_1\cdots\mu_n}\, J^{(m)}_{\mu_{n+1}\cdots\mu_m)}. \tag{27}$$

This satisfies the generalised Killing equation (10) because of Leibniz' rule.

1.6. EXAMPLE: FLAT SPACE

In order to illustrate the general formalism presented above, we consider the example of flat space, for which all conserved quantities of motion can be constructed explicitly. For

$$g_{\mu\nu}(x) = \delta_{\mu\nu}, \tag{28}$$

the Killing equation

$$R_{\mu,\nu} + R_{\nu,\mu} = 0 \tag{29}$$

has the general solution

$$R_\mu[a,\omega] = a_\mu + \omega_{\mu\nu}x^\nu, \quad \omega_{\mu\nu} = -\omega_{\nu\mu}. \tag{30}$$

Here a_μ and $\omega_{\mu\nu}$ are constant parameters labeling the various independent Killing vectors (ten in four dimensions). The constants of motion corresponding to these Killing vectors are:

$$J^{(1)}[a,\omega] = a_\mu p^\mu + \tfrac{1}{2}\omega_{\mu\nu}M^{\mu\nu}, \tag{31}$$

with $M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$. Clearly, the first term in eq. (31) generates a translation, whilst the second one generates Lorentz transformations.

At the level of second-rank tensors one finds—in addition to symmetrised products of Killing vectors—the universal solution $\delta_{\mu\nu}$ plus one new solution:

$$K_{\mu\nu}[\alpha,\beta] = \alpha\,\delta_{\mu\nu} + \beta\left(\delta_{\mu\nu}x^2 - x_\mu x_\nu\right). \tag{32}$$

These solutions correspond to the quadratic Casimir invariants of the Poincaré algebra:

$$J^{(2)}[\alpha,\beta] = \alpha p_\mu^2 + \tfrac{1}{2}\beta M_{\mu\nu}^2. \tag{33}$$

Since for scalar particles there exist no other independent higher-order Casimir invariants of the Poincaré algebra, all other Killing tensors can now be expressed as symmetrised products of the Killing vectors (30) and the second-rank Killing tensors (32). For example, the Killing tensors of rank 3 which can be constructed are of the form

$$\begin{aligned} L_{\mu\nu\lambda}[a^{(1)},\omega^{(1)}] &= \delta_{(\mu\nu}\left(a^{(1)}_{\lambda)} + \omega^{(1)}_{\lambda)\kappa}x^\kappa\right), \\ L_{\mu\nu\lambda}[a^{(2)},\omega^{(2)}] &= \left(\delta_{(\mu\nu}x^2 - x_{(\mu}x_\nu\right)\left(a^{(2)}_{\lambda)} + \omega^{(2)}_{\lambda)\kappa}x^\kappa\right). \end{aligned} \tag{34}$$

To these tensors correspond the Noether charges

$$J^{(3)}[a^{(i)},\omega^{(i)}] = p^2\left(a^{(1)}\cdot p + \tfrac{1}{2}\omega^{(1)}\cdot M\right) + \tfrac{1}{2}M^2\left(a^{(2)}\cdot p + \omega^{(2)}\cdot M\right). \tag{35}$$

Clearly, the constants of motion $(p_\mu, M_{\mu\nu})$ form the building blocks for the construction of the whole algebra, and the general form of the conserved charges is

$$\mathcal{J}(x,p) = \hat{c} + \hat{a}\cdot p + \tfrac{1}{2}\hat{\omega}\cdot M, \tag{36}$$

in which the co-efficients $(\hat{c},\hat{a},\hat{\omega})$ are arbitrary functions of the quadratic Casimir invariants (p^2, M^2).

2. Motions of spinning points in curved space–time

2.1. SPINNING SPACE

In this section we extend the spaces considered previously with additional fermionic dimensions, parametrised by vectorial Grassmann co-ordinates ψ^μ. Following refs. [5–10], we take the extension in such a way, that a supersymmetry is realised in these graded spaces; it acts on the co-ordinates as

$$\delta x^\mu = -\mathrm{i}\epsilon\psi^\mu, \qquad \delta\psi^\mu = \epsilon\dot{x}^\mu. \tag{37}$$

Such graded spaces were called spinning spaces in ref. [11]. An action for the extremal trajectories ("geodesics") of spinning space is

$$S = \int_1^2 \mathrm{d}\tau\left(\tfrac{1}{2}g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu + \frac{\mathrm{i}}{2}g_{\mu\nu}(x)\psi^\mu\frac{D\psi^\nu}{D\tau}\right), \tag{38}$$

where the covariant derivative of ψ^μ is defined by

$$D\psi^\mu/D\tau = \dot{\psi}^\mu + \dot{x}^\lambda\Gamma_{\lambda\nu}{}^\mu\psi^\nu. \tag{39}$$

Under a general variation of the co-ordinates $(\delta x^\mu, \delta\psi^\mu)$ the action changes by

$$\delta S = \int_1^2 \mathrm{d}\tau\left\{-\delta x^\mu\left(g_{\mu\nu}\frac{D^2x^\nu}{D\tau^2} + \frac{\mathrm{i}}{2}\psi^\kappa\psi^\lambda R_{\kappa\lambda\mu\nu}\dot{x}^\nu\right)\right.$$
$$\left. + \mathrm{i}\Delta\psi^\mu g_{\mu\nu}\frac{D\psi^\nu}{D\tau} + \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\delta x^\mu p_\mu - \frac{\mathrm{i}}{2}\delta\psi^\mu g_{\mu\nu}\psi^\nu\right)\right\}. \tag{40}$$

Here the canonical momentum is

$$p_\mu = g_{\mu\nu}\dot{x}^\nu - \tfrac{1}{2}\mathrm{i}\Gamma_{\mu\kappa\lambda}\psi^\kappa\psi^\lambda, \tag{41}$$

whilst $R_{\kappa\lambda\mu\nu}$ is the Riemann curvature tensor. Moreover, we have simplified the expression for δS by introducing a covariantised variation of ψ^μ:

$$\Delta\psi^\mu = \delta\psi^\mu + \delta x^\lambda\Gamma_{\lambda\nu}{}^\mu\psi^\nu. \tag{42}$$

The action S is stationary under arbitrary variations δx^μ and $\delta\psi^\mu$ vanishing at the end points if the following equations of motion are satisfied:

$$D^2x^\mu/D\tau^2 = -\tfrac{1}{2}\mathrm{i}\,\psi^\kappa\psi^\lambda R_{\kappa\lambda}{}^\mu{}_\nu\,\dot{x}^\nu, \tag{43}$$

$$D\psi^\mu/D\tau = 0. \tag{44}$$

We briefly consider the physical interpretation of these equations. The quantity

$$S^{\mu\nu} = -\mathrm{i}\psi^\mu\psi^\nu \tag{45}$$

can formally be regarded as the spin-polarisation tensor of the particle [5–8,12], and correspondingly eqs. (43), (44) describe the classical motion of a Dirac particle. Equation (44) implies that the spin tensor is covariantly constant. Equation (43) then becomes

$$D^2x^\mu/D\tau^2 = \tfrac{1}{2}S^{\kappa\lambda}R_{\kappa\lambda}{}^\mu{}_\nu\,\dot{x}^\nu. \tag{46}$$

It implies that there exist spin-dependent gravitational forces similar to the electro-magnetic Lorentz force,

$$\ddot{x}^\mu = (q/m)\,F^\mu_\nu\,\dot{x}^\nu, \tag{47}$$

with the spin-polarisation tensor replacing the scalar electric charge [12,13] (here for unit mass). Such forces in principle allow the determination of spin without any reference to the intrinsic electro-magnetic dipole moments which are associated with it for charged particles.

2.2. SYMMETRIES AND GENERALISED KILLING EQUATIONS

We now look for specific variations δx^μ and $\Delta\psi^\mu$ which leave the action off-shell invariant modulo boundary terms. We take the variations to be of the form

$$\delta x^\mu = \mathcal{R}^\mu(x,\dot{x},\psi) = R^{(1)\mu}(x,\psi) + \sum_{n=1}^{\infty}\frac{1}{n!}\dot{x}^{\nu_1}\cdots\dot{x}^{\nu_n}R^{(n+1)\mu}_{\nu_1\cdots\nu_n}(x,\psi),$$

$$\Delta\psi^\mu = \mathcal{S}^\mu(x,\dot{x},\psi) = S^{(0)\mu}(x,\psi) + \sum_{n=1}^{\infty}\frac{1}{n!}\dot{x}^{\nu_1}\cdots\dot{x}^{\nu_n}S^{(n)}_{\nu_1\cdots\nu_n}{}^\mu(x,\psi). \tag{48}$$

If the Lagrangian transforms into a total derivative

$$\delta S = \int_1^2 \mathrm{d}\tau\,\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\delta x^\mu p_\mu - \frac{\mathrm{i}}{2}\delta\psi^\mu g_{\mu\nu}\psi^\nu - \mathcal{J}(x,\dot{x},\psi)\right), \tag{49}$$

it follows that

$$\frac{\mathrm{d}\mathcal{J}}{\mathrm{d}\tau} = \mathcal{R}^\mu\left(g_{\mu\nu}\frac{D^2x^\nu}{D\tau^2} + \frac{\mathrm{i}}{2}\psi^\kappa\psi^\lambda R_{\kappa\lambda\mu\nu}\dot{x}^\nu\right) + \mathrm{i}\,\mathcal{S}^\mu g_{\mu\nu}\frac{D\psi^\nu}{D\tau}. \tag{50}$$

If the equations of motion are satisfied, the right-hand side vanishes and $\mathcal{J}$ is conserved. Again, this is Noether's theorem. Otherwise, expanding $\mathcal{J}(x, \dot{x}, \psi)$ in terms of the four-velocity,

$$\mathcal{J}(x, \dot{x}, \psi) = J^{(0)}(x, \psi) + \sum_{n=1}^{\infty} \frac{1}{n!} \dot{x}^{\mu_1} \cdots \dot{x}^{\mu_n} J^{(n)}_{\mu_1 \cdots \mu_n}(x, \psi), \tag{51}$$

and comparing the left- and right-hand sides of eq. (50) with the ansatz (48) for δx^{μ} and $\Delta \psi^{\mu}$, we find the following identities:

$$J^{(n)}_{\mu_1 \cdots \mu_n}(x, \psi) = R^{(n)}_{\mu_1 \cdots \mu_n}(x, \psi), \quad n \geq 1, \tag{52}$$

and

$$S^{(n)}_{\mu_1 \cdots \mu_n \nu}(x, \psi) = \mathrm{i} \frac{\partial J^{(n)}_{\mu_1 \cdots \mu_n}}{\partial \psi^{\nu}}(x, \psi), \quad n \geq 0. \tag{53}$$

Moreover, these quantities have to satisfy a generalisation of the Killing equations of the form [11]

$$J^{(n)}_{(\mu_1 \cdots \mu_n ; \mu_{n+1})} + \frac{\partial J^{(n)}_{(\mu_1 \cdots \mu_n}}{\partial \psi^{\sigma}} \Gamma_{\mu_{n+1})\kappa}{}^{\sigma} \psi^{\kappa} = \frac{\mathrm{i}}{2} \psi^{\kappa} \psi^{\lambda} R_{\kappa \lambda \nu (\mu_{n+1}} J^{(n+1)}{}_{\mu_1 \cdots \mu_n)}{}^{\nu}. \tag{54}$$

Writing as before $R^{(1)}_{\mu} = R_{\mu}$, $R^{(2)}_{\mu\nu} = K_{\mu\nu}$, $R^{(3)}_{\mu\nu\lambda} = L_{\mu\nu\lambda}$, etc., and $J^{(0)} = B$, this reduces for the lowest components to

$$\begin{aligned} B_{,\mu} + \frac{\partial B}{\partial \psi^{\sigma}} \Gamma_{\mu\kappa}{}^{\sigma} \psi^{\kappa} &= \frac{\mathrm{i}}{2} \psi^{\rho} \psi^{\sigma} R_{\rho\sigma\kappa\mu} R^{\kappa}, \\ R_{(\mu;\nu)} + \frac{\partial R_{(\mu}}{\partial \psi^{\sigma}} \Gamma_{\nu)\kappa}{}^{\sigma} \psi^{\kappa} &= \frac{\mathrm{i}}{2} \psi^{\rho} \psi^{\sigma} R_{\rho\sigma\kappa(\mu} K_{\nu)}{}^{\kappa}, \\ K_{(\mu\nu;\lambda)} + \frac{\partial K_{(\mu\nu}}{\partial \psi^{\sigma}} \Gamma_{\lambda)\kappa}{}^{\sigma} \psi^{\kappa} &= \frac{\mathrm{i}}{2} \psi^{\rho} \psi^{\sigma} R_{\rho\sigma\kappa(\mu} L_{\nu\lambda)}{}^{\kappa}, \quad \text{etc.} \end{aligned} \tag{55}$$

These equations have to hold independently of the equations of motion. The purely bosonic (ψ-independent) parts of these equations reduce to those we found for the scalar particle, eqs. (11)–(13). In particular, the bosonic terms in the Killing vectors R^{μ} define a Lie algebra by taking Lie derivatives as in eq. (20). Furthermore we note that, contrary to the bosonic case, the Killing scalar $B(x, \psi) = J^{(0)}(x, \psi)$ is not always an irrelevant constant, because it can depend non-trivially on x^{μ} and ψ^{μ}, as follows from the first of eqs. (55).

2.3. UNIVERSAL SOLUTIONS FOR SPINNING SPACE

In contrast to the scalar particle, the spinning particle admits several conserved quantities of motion in a general curved space–time with metric $g_{\mu\nu}(x)$ [11]. Specifically, we can construct the following four universal constants of motion:

1. Like in the bosonic case $g_{\mu\nu}$ itself is a Killing tensor:

$$K_{\mu\nu} = g_{\mu\nu}, \tag{56}$$

with all other Killing vectors and tensors (bosonic as well as fermionic) equal to zero. The corresponding constant of motion is the Hamiltonian

$$H(x,P) = \tfrac{1}{2}\, g^{\mu\nu}(x) P_\mu P_\nu, \tag{57}$$

where we have defined a covariant momentum

$$P_\mu = p_\mu + \tfrac{1}{2}\mathrm{i}\,\Gamma_{\mu\kappa\lambda}\,\psi^\kappa \psi^\lambda. \tag{58}$$

2. A second obvious solution is provided by the Grassmann-odd Killing vectors

$$R^\mu = \psi^\mu, \quad T_\mu^\nu = \mathrm{i}\delta_\mu^\nu. \tag{59}$$

Again all other Killing vectors and tensors are taken to vanish. This solution gives us the supercharge

$$Q = P_\mu \psi^\mu. \tag{60}$$

3. In addition to ordinary supersymmetry, the spinning particle action has a second non-linear supersymmetry, generated by Killing vectors

$$\begin{aligned} R_\mu &= \frac{-\mathrm{i}^{[d/2]}}{(d-1)!}\sqrt{-g}\,\varepsilon_{\mu\nu_1\cdots\nu_{d-1}}\,\psi^{\nu_1}\cdots\psi^{\nu_{d-1}}, \\ T_{\mu\nu} &= \frac{\mathrm{i}^{[(d-2)/2]}}{(d-2)!}\sqrt{-g}\,\varepsilon_{\mu\nu\nu_1\cdots\nu_{d-2}}\,\psi^{\nu_1}\cdots\psi^{\nu_{d-2}}. \end{aligned} \tag{61}$$

Obviously, the Grassmann parities of $(R_\mu, T_{\mu\nu})$ depend on the number of space–time dimensions. The corresponding constant of motion is the dual supercharge, given by

$$Q^* = \frac{-\mathrm{i}^{[d/2]}}{(d-1)!}\sqrt{-g}\,\varepsilon_{\mu_1\cdots\mu_d}\,P^{\mu_1}\psi^{\mu_2}\cdots\psi^{\mu_d}. \tag{62}$$

4. Finally, there exists a non-trivial Killing scalar

$$\Gamma_* \equiv J^{(0)} = -\frac{\mathrm{i}^{[d/2]}}{d!}\sqrt{-g}\,\varepsilon_{\mu_1\cdots\mu_d}\,\psi^{\mu_1}\cdots\psi^{\mu_d}. \tag{63}$$

This constant of motion acts as the Hodge star duality operator on ψ^μ. In quantum mechanics it becomes the γ^{d+1} element of the Dirac algebra. For this reason Γ_* is refered to as the chiral charge.

From the fundamental Dirac brackets

$$\begin{aligned} \{x^\mu, p_\nu\} &= \delta_\nu^\mu, \\ \{\psi^\mu, \psi^\nu\} &= -\mathrm{i}g^{\mu\nu}, \\ \{p_\mu, \psi^\nu\} &= \tfrac{1}{2}g^{\kappa\nu}g_{\kappa\lambda,\mu}\psi^\lambda, \\ \{p_\mu, p_\nu\} &= -\tfrac{1}{4}\mathrm{i}g^{\kappa\lambda}g_{\kappa\rho,\mu}g_{\lambda\sigma,\nu}\psi^\rho\psi^\sigma, \end{aligned} \tag{64}$$

we now find the following non-trivial Dirac brackets between these universal constants of motion:

$$\{Q,Q\} = -2\mathrm{i}\,H, \qquad \{Q,\Gamma_*\} = -\mathrm{i}Q^*. \tag{65}$$

Observe that $d = 2$ is an exceptional case: Q^* is linear and acts as an ordinary supersymmetry:

$$\{Q^*, Q^*\} = -2\mathrm{i}\,H, \qquad \{Q^*, \Gamma_*\} = -\mathrm{i}Q. \tag{66}$$

Hence in two dimensions the theory actually possesses an $N = 2$ supersymmetry. For $d \neq 2$, the right-hand side of eqs. (66) is to be replaced by zero.

3. Spinning particles in Schwarzschild space–time

3.1. CONSERVATION LAWS IN SCHWARZSCHILD SPACE–TIME

As an application of the generalised Killing equations for spinning space we discuss the motion of a spinning particle in a static and spherically symmetric gravitational field [#2]. The field is described by the Schwarzschild metric

$$\mathrm{d}s^2 = -\left(1 - \frac{\alpha}{r}\right)\mathrm{d}t^2 + \frac{1}{(1-\alpha/r)}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right), \tag{67}$$

where $\alpha = 2MG$, M being the total mass of the spherically symmetric object in the centre of the field. It is well known, that the Schwarzschild metric possesses four Killing vector fields of the form

$$D^{(\alpha)} \equiv R^{(\alpha)\mu}(x)\partial_\mu, \quad \alpha = 0, \ldots, 3, \tag{68}$$

where

$$\begin{aligned} D^{(0)} &= \partial/\partial t, \\ D^{(1)} &= -\sin\varphi\,\partial/\partial\theta + \cot\theta\cos\varphi\,\partial/\partial\varphi, \\ D^{(2)} &= \cos\varphi\,\partial/\partial\theta - \cot\theta\sin\varphi\,\partial/\partial\varphi, \\ D^{(3)} &= \partial/\partial\varphi. \end{aligned} \tag{69}$$

These Killing vector fields express the time-translation invariance and the rotation symmetry of the gravitational field. They generate the corresponding Lie algebra o(1,1) × so(3):

$$\begin{aligned} \left[D^{(i)}, D^{(j)}\right] &= -\varepsilon^{ijk}D^{(k)}, \\ \left[D^{(0)}, D^{(i)}\right] &= 0. \end{aligned} \tag{70}$$

The first generalised Killing equation (55) shows that with each Killing vector $R_\mu^{(\alpha)}$ there is associated a Killing scalar $B^{(\alpha)}$. These Killing scalars are necessary to obtain the constants of motion

$$J^{(\alpha)} = B^{(\alpha)} + m\dot{x}^\mu R_\mu^{(\alpha)}. \tag{71}$$

The Killing scalars have a natural interpretation: the constants of motion represent the *total* angular momentum, which is the sum of the orbital and the spin

#2 More details can be found in ref. [14].

angular momentum. In general, orbital angular momentum is no longer separately conserved; therefore the Killing vector itself does *not* give a conserved quantity of motion. The contribution of spin is contained in the Killing scalars, and has to be added.

Inserting the expression for the connections and the Riemann curvature components of the Schwarzschild space–time in eq. (55), we obtain for the Killing scalars

$$
\begin{aligned}
B^{(0)} &= (-\mathrm{i}\alpha/2r^2)\ \psi^t\psi^r,\\
B^{(1)} &= \mathrm{i}r\sin\varphi\ \psi^r\psi^\theta + \mathrm{i}r\sin\theta\cos\theta\cos\varphi\ \psi^r\psi^\varphi - \mathrm{i}r^2\sin^2\theta\cos\varphi\ \psi^\theta\psi^\varphi,\\
B^{(2)} &= -\mathrm{i}r\cos\varphi\ \psi^r\psi^\theta + \mathrm{i}r\sin\theta\cos\theta\sin\varphi\ \psi^r\psi^\varphi - \mathrm{i}r^2\sin^2\theta\sin\varphi\ \psi^\theta\psi^\varphi,\\
B^{(3)} &= -\mathrm{i}r\sin^2\theta\ \psi^r\psi^\varphi - \mathrm{i}r^2\sin\theta\cos\theta\ \psi^\theta\psi^\varphi.
\end{aligned}
\tag{72}
$$

Upon subtitution in $J^{(\alpha)}$, eq. (71), and using the spin-tensor notation introduced in eq. (45), one finds

$$
\begin{aligned}
J^{(0)} &\equiv E = m\left(1-\frac{\alpha}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau} - \frac{\alpha}{2r^2}S^{rt},\\
J^{(1)} &= -r\sin\varphi\left(mr\frac{\mathrm{d}\theta}{\mathrm{d}\tau} + S^{r\theta}\right) - \cos\varphi\left(\cot\theta\, J^{(3)} - r^2S^{\theta\varphi}\right),\\
J^{(2)} &= r\cos\varphi\left(mr\frac{\mathrm{d}\theta}{\mathrm{d}\tau} + S^{r\theta}\right) - \sin\varphi\left(\cot\theta\, J^{(3)} - r^2S^{\theta\varphi}\right),\\
J^{(3)} &= r\sin^2\theta\left(mr\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} + S^{r\varphi}\right) + r^2\sin\theta\cos\theta\, S^{\theta\varphi}.
\end{aligned}
\tag{73}
$$

In addition to these constants of motion, the universal conserved charges such as the world-line Hamiltonian and supercharge also provide information about the allowed orbits of the particle. Specifically, we consider motions for which

$$H = -\tfrac{1}{2}m^2c^2, \quad Q = 0. \tag{74}$$

The first equation implies geodesic motion (equality of proper time with the geodesic interval):

$$\mathrm{d}\tau^2 = -\mathrm{d}s^2, \tag{75}$$

cf. eqs. (2), (67). The second equation expresses the fact that spin represents only three independent degrees of freedom. Indeed, we can now solve for ψ^t in terms of the spatial components ψ^i:

$$\left(1-\frac{\alpha}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau}\psi^t = \frac{1}{(1-\alpha/r)}\frac{\mathrm{d}r}{\mathrm{d}\tau}\psi^r + r^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\psi^\theta + \sin^2\theta\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\psi^\varphi\right). \tag{76}$$

As a result, the (classical) chiral charge Γ_* vanishes as well.

From eqs. (73) one can derive a useful identity

$$r^2\sin\theta\, S^{\theta\varphi} = J^{(1)}\sin\theta\cos\varphi + J^{(2)}\sin\theta\sin\varphi + J^{(3)}\cos\theta. \tag{77}$$

In physical terms, this identity simply states that there is no orbital angular momentum in the radial direction.

Combining eqs. (73), (74) one obtains a complete set of first integrals of motion, expressing the velocities as functions of the co-ordinates, the spatial spin components and the constants of motion:

$$\begin{aligned}
\frac{dt}{d\tau} &= \frac{1}{(1-\alpha/r)}\left(\frac{E}{m} + \frac{\alpha}{2mr^2}S^{rt}\right), \\
\frac{dr}{d\tau} &= \left[\left(1-\frac{\alpha}{r}\right)^2\left(\frac{dt}{d\tau}\right)^2 - 1 + \frac{\alpha}{r}\right. \\
&\qquad \left. -r^2\left(1-\frac{\alpha}{r}\right)\left\{\left(\frac{d\theta}{d\tau}\right)^2 + \sin^2\theta\left(\frac{d\varphi}{d\tau}\right)^2\right\}\right]^{1/2}, \\
\frac{d\theta}{d\tau} &= \frac{1}{mr^2}\left(-J^{(1)}\sin\varphi + J^{(2)}\cos\varphi - rS^{r\theta}\right), \\
\frac{d\varphi}{d\tau} &= \frac{1}{mr^2\sin^2\theta}J^{(3)} - \frac{1}{mr}S^{r\varphi} - \frac{1}{m}\cot\theta\, S^{\theta\varphi},
\end{aligned} \tag{78}$$

in which

$$S^{rt} = \frac{mr^2}{E}\left(\frac{d\theta}{d\tau}S^{r\theta} + \sin^2\theta\frac{d\varphi}{d\tau}S^{r\varphi}\right). \tag{79}$$

Finally, the rate of change of the spins is determined by

$$\begin{aligned}
\frac{d\psi^r}{d\tau} &= r\left(1-\frac{3\alpha}{2r}\right)\left(\frac{d\theta}{d\tau}\psi^\theta + \sin^2\theta\frac{d\varphi}{d\tau}\psi^\varphi\right), \\
\frac{d\psi^\theta}{d\tau} &= -\frac{1}{r}\left(\frac{dr}{d\tau}\psi^\theta + \frac{d\theta}{d\tau}\psi^r\right) + \sin\theta\cos\theta\frac{d\varphi}{d\tau}\psi^\varphi, \\
\frac{d\psi^\varphi}{d\tau} &= -\left(\frac{1}{r}\frac{dr}{d\tau} + \cot\theta\frac{d\theta}{d\tau}\right)\psi^\varphi - \frac{1}{r}\frac{d\varphi}{d\tau}\psi^r - \cot\theta\frac{d\varphi}{d\tau}\psi^\theta.
\end{aligned} \tag{80}$$

Equations (78)–(80) can be integrated to give the full solution of the equations of motion for all co-ordinates and spins.

3.2. SPECIAL SOLUTIONS

As an application of the results obtained in section 3.1 we study the special case of motion in a plane, for which we choose $\theta = \pi/2$. Unlike for scalar point particles, this is not the generic case, because in general orbital angular momentum is not conserved separately.

For spinning particles, motion in a plane occurs in two kinds of situation. The first possibility is radial motion, for which $\dot{\varphi} = 0$. In this case there is no orbital angular momentum and spin is conserved independently. Clearly, either the particle escapes to infinity or hits the centre of the potential after a finite time.

The second possibility concerns motion for which $\dot{\varphi} \neq 0$. In this case orbital and spin angular momentum decouple if they are parallel. Hence we impose the conditions

$$S^{\theta\varphi} = 0, \quad S^{r\theta} = 0, \tag{81}$$

from which it now follows that $J^{(1),(2)}$ vanish. In this case we have two constants of motion:

$$L = mr^2\dot{\varphi}, \quad \Sigma \equiv J^{(3)} - L = rS^{r\varphi}. \tag{82}$$

From the first of eqs. (78) we now find a formula for the gravitational redshift given by

$$\mathrm{d}t = \frac{\mathrm{d}\tau}{(1-\alpha/r)}\left(\frac{E}{m} + \frac{\alpha}{2mEr^3}L\Sigma\right). \tag{83}$$

The first term corresponds to the usual time dilation in a gravitational field for spinless particles. In this case, there is an additional contribution from spin–orbit coupling. This shows that time dilation is not a purely geometrical effect, but also has a dynamical component [12].

From the second of eqs. (78) and the first of eqs. (82) we obtain the equation for the orbit of the particle:

$$\frac{1}{r}\frac{\mathrm{d}r}{\mathrm{d}\varphi} = \sqrt{\frac{(E^2-m^2)}{L^2}r^2 - 1 + \frac{m\alpha}{L}\left(\frac{mr}{L} + \frac{J^{(3)}}{mr}\right)}. \tag{84}$$

In terms of dimensionless quantities

$$\begin{aligned} \epsilon &= E/m, \quad x = r/\alpha, \\ \ell &= L/m\alpha, \; \Delta = \Sigma/L, \end{aligned} \tag{85}$$

we find for the stationary points x_m of the orbit, as defined by the vanishing of the right-hand side of eq. (84):

$$\epsilon^2 = 1 - \frac{1}{x_m} + \frac{l^2}{x_m^2} - \frac{l^2}{x_m^3}(1+\Delta). \tag{86}$$

Of course, all calculations presented here are rather formal, as Δ is not a pure number. However, the equations might be applicable to realistic physics situations if it is allowed to replace Δ in certain limiting cases by a real number. Since the pseudo-classical equations acquire physical meaning when averaged over in functional integrals [5,15] (i.e. in the path integral of the quantum Dirac particle), such a limit might arise in the semi-classical regime of the quantum theory, as implied by the correspondence principle. In the following we assume that such a numerical value of Δ has been obtained and leads to valid results, at least in expansions to first order in Δ (where the fact that $\Delta^2 = 0$ plays no role).

For $\epsilon \geq 1$ we have open orbits for which there is at most one point of closest approach, the perihelion. If for fixed l the energy exceeds a critical value, the particle can cross into the central region of the potential ($x < 1$). The critical value is given by

$$\epsilon^2_{\rm crit} = \epsilon^2(x_m^{(-)}), \tag{87}$$

where

$$x_m^{(-)} = l^2 \left(1 - \sqrt{1 - \frac{3(1+\Delta)}{l^2}} \right). \tag{88}$$

For $\epsilon < 1$ there are bound states corresponding to quasi-periodic orbits, which have both a perihelion and an aphelion. The stationary points of the orbit are again determined by [#3] eq. (86). Solutions of this equation exist only for

$$l^2 \geq 3\,(1+\Delta). \tag{89}$$

In particular, for circular orbits there exists a minimal radius given by

$$x_m = l^2 = 3\,(1+\Delta). \tag{90}$$

For this critical orbit the energy is to first order in Δ:

$$\epsilon^2_{\rm crit} = \tfrac{1}{9}\,(8+\Delta), \tag{91}$$

whilst the time-dilation factor in the critical orbit is given by

$$({\rm d}t/{\rm d}\tau)_{\rm crit} = \sqrt{2}\left(1 - \tfrac{3}{8}\Delta\right). \tag{92}$$

For non-circular motion one finds that the perihelion of the orbit precesses as for a spinless particle, but at a different, spin dependent rate. In the weak-coupling limit (slow precession of the perihelion), we obtain for the precession angle after one period

$$\Delta\varphi = \frac{3\pi\alpha}{k}\,(1+\Delta), \tag{93}$$

where k is the *semilatus rectum* of the elliptical orbit, and we have neglected terms of $O(k^{-2})$. Hence in this approximation spin-dependent effects disappear for $\Delta = O(k^{-p})$ whenever $p \geq 1$. Note, that spin-dependent gravitational effects can be larger or smaller than for a spinless particle depending on the sign of Δ, i.e. the relative orientation of L and Σ. This we interpret as a classical analogue of fine splitting.

As a final remark, we observe that even if an a priori numerical value for Δ cannot be assigned, its appearance in various places like in eqs. (91)–(93) still allows the pseudo-classical theory to make quantitative predictions by comparing different physical processes in the regime where the semi-classical limit applies. For a discussion of some related issues in the case of spinning particles in electromagnetic fields we also refer to ref. [12].

[#3] For the special case of circular motion, this equation determines the radius of the orbit, which now is of course independent of φ.

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